Unconstrained Variational Statements for Initial and Boundary-Value Problems

T. E. Simkins*
U.S. Army Armament Research and Development Command, Benet Weapons Laboratory, LCWSL, Watervliet, N.Y.

A procedure is developed for generating variational statements suitable for obtaining approximate solutions to boundary-initial value problems. When applied to the Euler differential equations governing the behavior of an elastic body, the procedure leads directly to a modification of Hamilton's principle first given by H. F. Tiersten, who also demonstrated its utility in solving problems involving surfaces of discontinuity. The essence of the procedure is to introduce all boundary and initial conditions into the variational statement as natural boundary conditions. This is accomplished through the use of Lagrange multipliers, in which all initial condition terms as well as boundary terms are determined analytically. The result is a variational statement in which completely unconstrained trial functions may be assumed as a basis for an approximate solution. Several applications are given, including the response of a beam subject to a moving concentrated mass loading.

I. Introduction

THEOREMS establishing the correspondence between certain boundary value problems and the calculus of variations appear in detail in the treatise of Collatz.[1] More recently, Rund[2] has produced more general theorems through the use of the transversality conditions from which the theorems of Collatz emerge as special cases. Neither of these works, however, attempts to establish variational statements for the solution of initial value problems, which is the subject of the work herein.

Recent work by Bailey[3,4] has shown that Hamilton's law of varying action is capable of yielding approximate solutions to initial and boundary-initial value problems. The variational form used by Bailey allows the function and its derivative to vary at the upper limit of integration of the time interval. At the lower limit, these quantities are constrained to satisfy specified initial conditions. It appears that a more general method would free the variations at the lower limit as well, thus broadening the class of admissible trial functions. Although much has been written[5] on the subject of removal of constraints on the boundary variations, the applications usually deal with elliptic rather than hyperbolic systems, where it is customary to introduce the constraints of the problem as natural boundary conditions. The constraints themselves thus become subject to approximation through the variational process. Convergence, when achieved, tends toward a solution satisfying these constraint conditions. When all of the constraints of the problem are introduced in this manner, the result is a completely unconstrained variational statement, whereby the trial functions need not identically satisfy any boundary conditions. As the use of Lagrange multipliers in freeing boundary variations of constraints is by now classical, this work is fundamentally concerned with extending the method to remove constraining time conditions.

The Lagrange multiplier procedure adds each constraint as a zero times a Lagrange multiplier to the previously unconstrained variational statement. In this way, each constraint is made to appear as a natural boundary condition and, in some cases where a functional exists (i.e., variational "principles"), it may be modified to include these terms. The multipliers can usually be identified in terms of values of the function and its derivatives on the bounding surface of the domain of integration. The act of freeing the boundary variations will not result in the loss of a variational "principle" provided the constraint is holonomic, i.e., a functional will still exist though modified by additive products of the Lagrange multipliers times the individual constraint relations. On the other hand, should any of the constraints be nonholonomic, the existence of a functional is denied and one has in its place a less elegant variational "statement" which may nevertheless provide a basis for an approximate solution to the problem at hand.

In spite of the apparent generality of the Lagrange multiplier method, its application to Hamilton's principle (a *constrained* variational principle) for the solution of initial value problems is not obvious. Indeed, Bailey found it fruitful to employ Hamilton's law of varying action in which no functional exists. Once the quest of a functional is abandoned, however, unconstrained variational formulations for initial value problems are immediately possible, as was first shown by Tiersten.[6] Since the purpose of Tiersten's work at the time did not involve explicit solutions in the time domain, the success of his method for achieving solutions to initial value problems was never tested. Further, Tiersten's procedure requires a special introduction of one of the initial conditions into the variational statement, making incomplete use of the Lagrange multiplier method in the time domain. In what follows, the variational statement first given by Tiersten will be constructed systematically from the given differential equation, boundary, and initial conditions in a manner similar to the Galerkin procedure.[7] Solutions to the free oscillator, the two-dimensional wave equation, and the motion of a beam to a moving concentrated mass are offered as evidence of success of the method.

The variational form presented herein does not produce a functional. Wu,[8] however, has successfully formulated initial and boundary-initial value problems using an unconstrained adjoint variational principle in which a functional is indeed produced. Wu's treatment requires replacing the given boundary and initial conditions with a set of artificial conditions containing a parameter that eventually is allowed to become very large numerically. Using finite element approximations, Wu was able to achieve excellent agreement with the exact solutions of several partial differential equations in one and two dimensions.

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*Research Mechanical Engineer.

II. Variational Statements

The general procedure to be followed in constructing unconstrained variational statements will first be shown by treating the simple boundary value problem:

$$y'' - q(x) = 0 \tag{1a}$$

$$y'(0) = \beta \qquad y(1) = 0 \tag{1b}$$

According to a theorem of Collatz,[1] the first of the boundary conditions, Eq. (1b), is suppressible, leading to the following variational formulation:

$$\int_0^1 [y'\delta y' + q\delta y]\mathrm{d}x + \beta\delta y(0) = 0 \tag{2}$$

The boundary value $y'(0) = \beta$ has been brought into the variational formulation directly, and assumed trial functions for the solution of Eq. (2) need not satisfy this constraint identically, i.e., insofar as this boundary condition is concerned the trial functions are unconstrained. A formal procedure is desired for generating Eq. (2) from the system (1). As in Galerkin's procedure, the differential equation is multiplied by δy and integrated over the domain (0,1). Any boundary condition can then be added to this result as a zero times a Lagrange multiplier to be determined. Thus,

$$0 = \int_0^1 [y'' - q(x)]\delta y \mathrm{d}x + (y'(0) - \beta)\delta\lambda$$

Integrating by parts and substituting the second of the boundary conditions, Eq. (1b) yields:

$$0 = -y'(0)\delta y(0) + y'(0)\delta\lambda - \beta\delta\lambda - \int_0^1 (y'\delta y' + q\delta y)\mathrm{d}x \tag{3}$$

Equation (3) reduces to Eq. (2) only if $\delta\lambda \equiv \delta y(0)$, i.e., redundant boundary terms are eliminated through appropriately defining $\delta\lambda$. This is the general procedure to be followed. It follows that all boundary conditions can be brought into the variational formulation by this procedure. A completely unconstrained variational statement will then result, whereby trial functions can be assumed which do not have to satisfy any boundary conditions. It remains to be demonstrated that the procedure can be extended to achieve formulations of initial value problems as well. The boundary conditions, Eq. (1b), are therefore replaced by the conditions:

$$y(0) = \alpha \qquad y'(0) = \beta \tag{1c}$$

The variational statement which follows is simply

$$\int_0^1 (y'' - q(x))\delta y \mathrm{d}x + (y(0) - \alpha)\delta\lambda_1 + (y'(0) - \beta)\delta\lambda_2 = 0 \tag{4}$$

After an integration by parts

$$y'(1)\delta y(1) - y'(0)\delta y(0) + (y(0) - \alpha)\delta\lambda_1 + (y'(0) - \beta)\delta\lambda_2 - \int_0^1 (y'\delta y' + q\delta y)\mathrm{d}x = 0 \tag{5}$$

Defining $\delta\lambda_2 = \delta y(0)$ eliminates the redundant boundary terms at the lower limit $x = 0$, but no similar definition for $\delta\lambda_1$ is obvious at this stage. A second integration by parts, however, assists in solving this problem. Thus

$$\int_0^1 (y\delta y'' - q\delta y)\mathrm{d}x + (y(0) - \alpha)\delta\lambda_1 - \beta\delta y(0) + y'(1)\delta y(1) - y(1)\delta y'(1) + y(0)\delta y'(0) = 0 \tag{6}$$

The redundant boundary terms at the lower limit of integration are again eliminated through suitable definition of $\delta\lambda$, i.e., $\delta\lambda_1 = -\delta y'(0)$, whereupon the variational statement Eq. (5), corresponding to this initial value problem becomes

$$y'(1)\delta y(1) - (y(0) - \alpha)\delta y'(0) - \beta\delta y(0) - \int_0^1 (y'\delta y' + q\delta y)\mathrm{d}x = 0 \tag{7}$$

Equation (7) may be solved (approximately) using trial functions which are completely unconstrained.

III. Extended Hamilton's Principle

To add more generality to the problems considered and to show that the procedure that has been constructed will indeed yield the variational formulation achieved by Tiersten as an extension of Hamilton's principle, the Euler differential equations for an elastic body are next considered. In the following, each index ranges from one to three, unless otherwise indicated, and a repeated index denotes summation. Dot and comma notation are also employed. The Euler equations are:

$$\frac{\partial U}{\partial u_k} + \rho\ddot{u}_k - G_{\ell k,\ell} = 0 \tag{8}$$

where U represents the total potential energy function, and $G_{\ell k}$ the stress tensor. $u_k(x_i,t)$ represents the elastic displacement from equilibrium, and ρ the mass density of the material. The boundary conditions (constraints) prescribe the functions u_i on portions of the surface denoted by S_c, while on remaining portions S_N of the boundary, the u_i are unknown but the quantities

$$F_k \equiv n_\ell \frac{\partial U}{\partial u_{k,\ell}} \equiv n_\ell G_{\ell k}$$

are prescribed functions of x_j and t. n_ℓ denotes components of the outward-directed unit normal to the surface bounding the region V. The function u and its time derivative are specified at time t_0. Letting barred quantities denote specified values:

$$u_k - \bar{u}_k = 0 \quad \text{on} \quad S_c$$

$$-n_\ell G_{\ell k} + \bar{F}_k = 0 \quad \text{on} \quad S_N$$

$$u_k - \bar{u}_k(t_0) = 0, \quad \dot{u}_k - \bar{v}_k = 0; \quad t = t_0 \tag{9}$$

The differential Eq. (8) is multiplied by δu_k and integrated over the space-time domain. Each constraint is added to the result as a zero times a Lagrange multiplier. Hence,

$$\begin{aligned} 0 = &\int_{t_0}^{t_1} \int_V \left(\frac{\partial U}{\partial u_k} + \rho\ddot{u}_k - G_{\ell k,\ell}\right)\delta u_k \mathrm{d}V\mathrm{d}t \\ &+ \int_{t_0}^{t_1} \mathrm{d}t \int_{S_N} (n_\ell G_{\ell k} - \bar{F})\delta\lambda_k^{(1)}\mathrm{d}S \\ &+ \int_{t_0}^{t_1} \mathrm{d}t \int_{S_c} (u_k - \bar{u}_k)\delta\lambda_k^{(2)}\mathrm{d}S \\ &+ \int_V [u_k(t_0) - \bar{u}_k(t_0)]\delta\lambda_k^{(3)}\mathrm{d}V \\ &+ \int_V (\dot{u}_k(t_0) - \bar{v}_k)\delta\lambda_k^{(4)}\mathrm{d}V \end{aligned} \tag{10a}$$

Using the divergence theorem and an integration by parts in the time domain yields a variational form from which some of the $\delta\lambda$ quantities can be defined:

$$
\begin{aligned}
0= & \int_{t_0}^{t_1} \int_V \left(\frac{\partial U}{\partial u_k} \delta u_k + G_{\ell k} \delta u_{k,\ell} - \rho \dot{u}_k \delta \dot{u}_k \right) \mathrm{d}V \mathrm{d}t \\
& + \int_{t_0}^{t_1} \mathrm{d}t \int_{S_N} \left(n_\ell G_{\ell k} \delta\lambda_k^{(1)} - \bar{F} \delta\lambda_k^{(1)} - n_\ell G_{\ell k} \delta u_k \right) \mathrm{d}S \\
& + \int_{t_0}^{t_1} \mathrm{d}t \int_{S_c} \left[(u_k - \bar{u}_k) \delta\lambda_k^{(2)} - n_\ell G_{\ell k} \delta u_k \right] \mathrm{d}S \\
& + \int_V \{ \rho \dot{u}_k (t_1) \delta u_k (t_1) - \rho \dot{u}_k (t_0) \delta u_k (t_0) \\
& + (u_k (t_0) - \bar{u}_k (t_0)) \delta\lambda_k^{(3)} + (\dot{u}_k (t_0) - \bar{v}_k) \delta\lambda_k^{(4)} \} \mathrm{d}V
\end{aligned} \tag{10b}
$$

In accordance with the procedure previously demonstrated

$$
\delta\lambda_k^{(1)} = \delta u_k \quad \text{on } S_N, \quad \delta\lambda_k^{(4)} = \rho \delta u_k (t_0) \tag{11}
$$

The remaining $\delta\lambda$ quantities can be determined by a second application of the divergence theorem and integration by parts in time. To avoid a mere retracing of steps during this process, it is necessary to employ the following relation which is valid for an elastic solid:

$$
\int_V G_{\ell k} \delta u_{k,\ell} \mathrm{d}V = \int_V \delta G_{\ell k} u_{k,\ell} \mathrm{d}V \tag{12}
$$

Substituting Eqs. (11) and (12) into Eq. (10b), and performing the required integrations:

$$
\begin{aligned}
0= & \int_{t_0}^{t_1} \int_V \left(\frac{\partial U}{\partial u_k} \delta u_k - u_k \delta G_{\ell k,\ell} + \rho u_k \delta \ddot{u}_k \right) \mathrm{d}V \mathrm{d}t \\
& + \int_{t_0}^{t_1} \int_{S_N} \left(-\bar{F} \delta u_k + n_\ell \delta G_{\ell k} u_k \right) \mathrm{d}S \\
& + \int_{t_0}^{t_1} \int_{S_c} \{ u_k - \bar{u}_k) \delta\lambda_k^{(2)} - n_\ell G_{\ell k} \delta u_k + n_\ell \delta G_{\ell k} u_k \} \mathrm{d}S \\
& + \int_V \{ \rho \dot{u}_k (t_1) \delta u_k (t_1) + [u_k (t_0) - \bar{u}_k (t_0)] \delta\lambda_k^{(3)} \\
& - \rho \bar{v}_k \delta u_k (t_0) - \rho u_k (t_0) \delta \dot{u}_k (t_1) + \rho u_k (t_0) \delta \dot{u}_k (t_0) \} \mathrm{d}V
\end{aligned} \tag{10c}
$$

The definitions thus obtained are:

$$
\begin{aligned}
\delta\lambda_k^{(2)} &= - n_\ell \delta G_{\ell k} \quad \text{on } S_c \\
\delta\lambda_k^{(3)} &= - \rho \delta \dot{u}_k (t_0)
\end{aligned} \tag{13}
$$

Any one of Eqs. (10a-c) may be used as a variational formulation to achieve approximate solutions to the system of Eqs. (8) and (9). For most work it is desirable that the trial functions be as simple as possible and, in view of the lower order of derivatives present, Eq. (10b) would most likely be

Table 1 Solution to wave equation $0 \le x \le 2.0$; $0 \le t \le 2.0$ (exact values in parentheses)

t/x	0.0	0.4	0.8	1.0	1.2	1.6	2.0
0.0	.000060	.587782	.951066	1.000000	.951063	.587785	.000056
	(.000000)	(.588785)	(.951057)	(1.000000)	(.951057)	(.587785)	(.000000)
0.4	.000014	.475532	.769422	.809006	.769421	.475533	.000013
	(.000000)	(.475528)	(.769421)	(.809017)	(.769421)	(.475528)	(.000000)
0.8	−.000001	.181636	.293889	.309010	.293890	.181635	.000001
	(.000000)	(.181636)	(.293893)	(.309017)	(.293893)	(.181636)	(.000000)
1.0	−.000003	−.000000	−.000002	−.000001	−.000001	−.000001	−.000002
	(.000000)	(.000000)	(.000000)	(.000000)	(.000000)	(.000000)	(.000000)
1.2	−.000000	−.181637	−.293889	−.309010	−.293890	−.181636	−.000001
	(.000000)	(−.181636)	(−.293893)	(−.309017)	(−.293893)	(−.181636)	(−.000000)
1.6	−.000013	−.475531	−.769420	−.809006	−.769420	−.475531	−.000012
	(.000000)	(−.475528)	(−.769421)	(−.809017)	(−.769421)	(−.475528)	(−.000000)
2.0	−.000036	−.587786	−.951055	−.999986	−.951056	−.587785	−.000033
	(.000000)	(−.587785)	(−.951057)	(−1.000000)	(−.951057)	(−.587785)	(−.000000)

Table 2 Solution to wave equation $0 \le x \le 2.0$; $0 \le t \le 4.0$ (exact values in parentheses)

t/x	0.0	0.4	0.8	1.0	1.2	1.6	2.0
0.0	.000037	.595863	.964161	1.013774	.964161	.595867	.000031
	(.000000)	(.587785)	(.951057)	(1.000000)	(.951057)	(.587785)	(.000000)
0.8	.000008	.183947	.297637	.312952	.297638	.183948	.000007
	(.000000)	(.181636)	(.293893)	(.309017)	(.293893)	(.181636)	(.000000)
1.6	−.000014	−.477206	−.772139	−.811866	−.772139	−.477207	−.000012
	(.000000)	(−.475528)	(−.769421)	(−.809017)	(−.769421)	(−.475528)	(−.000000)
2.0	−.000013	−.587942	−.951308	−1.000253	−.951309	−.587943	−.000013
	(.000000)	(−.587785)	(−.951057)	(−1.000000)	(−.951057)	(−.587785)	(−.000000)
2.4	−.000006	−.474475	−.767707	−.807204	−.767707	−.474475	−.000007
	(.000000)	(−.475528)	(−.769421)	(−.809017)	(−.769421)	(−.475528)	(−.000000)
3.2	−.000001	.181191	.293165	.308247	.293166	.181191	.000001
	(.000000)	(.181636)	(.293893)	(.309017)	(.293893)	(.181636)	(.000000)
4.0	.000002	.587756	.951014	.999945	.951014	.587755	.000005
	(.000000)	(.587785)	(.951057)	(1.000000)	(.951057)	(.587785)	(.000000)

chosen as the variational formulation for the problem. On the other hand, it is sometimes convenient to use trial functions which are known to satisfy the differential equation, letting the variational process take care of the fit at the boundaries. For this purpose Eq. (10a) is most suitable; it is doubtful if Eq. (10c) has any computational value over the other two. Substituting Eqs. (11) and (13) into Eq. (10b):

$$0=\int_{t_0}^{t_1}\left[\int_V \delta L \mathrm{d}V+\int_{S_N}\bar{F}\delta u_k \mathrm{d}S\right.$$

$$\left.+\int_{S_c}\delta\{n_\ell G_{\ell k}(u_k-\bar{u}_k)\}\mathrm{d}S\right]\mathrm{d}t$$

$$+\int_V \mathrm{d}V\{-\dot{u}_k(t_1)\delta u_k(t_1)+\bar{v}_k\delta u_k(t_0)$$

$$+[u_k(t_0)-\bar{u}_k(t_0)]\}\rho\delta\dot{u}_k(t_0) \tag{14}$$

where L is the Lagrange density

$$L=\tfrac{1}{2}\rho\dot{u}_k\dot{u}_k-U(u_k,u_{k,\ell},x_j)$$

Equation (14) and the result obtained by Tiersten are identical except that in the interest of simplicity, no material surface of discontinuity has been considered in Eq. (14). It is to be noted that all variations are unconstrained so that the trial functions used in seeking an approximate solution need not satisfy any boundary or initial conditions a priori. If trial functions can be chosen that do satisfy some of the boundary constraints beforehand, convergence will usually be more rapid.

IV. Applications

Example 1: Wave Equation

$$S\frac{\partial^2 u}{\partial x^2}-\rho\frac{\partial^2 u}{\partial t^2}=0$$

$$u(0,t)=g_0(t),\quad u(\ell,t)=g_1(t)$$

$$u(x,0)=h_0(x),\quad \dot{u}(x,0)=h_1(x) \tag{15}$$

Thus,

$$U=\tfrac{1}{2}S(u')^2 \qquad \bar{F}\equiv 0$$

$$G=\frac{\partial U}{\partial u'}=Su'$$

Substituting the boundary conditions and the expressions for U and G into Eq. (14) results in the following variational statement:

$$0=\int_0^{t_1}\left\{\int_0^{\ell}(a^2\dot{u}\delta\dot{u}-u'\delta u')\mathrm{d}x+[u(\ell,t)\right.$$

$$-g_1(t)]\delta u'(\ell,t)-[u(0,t)-g_0(t)]\delta u'(0,t)+u'(\ell,t)\delta u(\ell,t)$$

$$\left.-u'(0,t)\delta u(0,t)\right\}\mathrm{d}t+\int_0^{\ell}a^2[-\dot{u}(x,t_1)\delta u(x,t_1)$$

$$+h_1(x)\delta u(x,0)+[u(x,0)-h_0(x)]\delta\dot{u}(x,0)]\mathrm{d}x \tag{16}$$

where $a^2=\rho/S$. A matrix formulation of Eq. (16) is achieved by substituting the approximation

$$u(x,t)=\sum_{j=1}^{N\times N}a_j(x,t)c_j$$

as in the Ritz procedure but without any constraint requirements on the a_j set a priori. The result is a set of algebraic equations of the form

$$\sum_{j=1}^{N\times N}k_{ij}c_j=r_i \quad i=1,N\times N \tag{17}$$

for the determination of the constants $c_j(\ell,t_1)$. Results are given in Tables 1 and 2 for the case:

$$g_0=g_1=h_1=0;\quad h_0(x)=\sin\pi x/\ell;\quad \ell=2$$

The shape functions $a_j(x,t)$ are taken to be products of polynomials in x and t. Good convergence is obtained for $N=8$. As expected, Tables 1 and 2 show a decline in accuracy as the interval of integration is doubled.

Example 2. Free Oscillator–Particle Mechanics

$$\ddot{u}+\omega^2u=0,\quad u(0)=u_0,\quad \dot{u}(0)=v_0 \tag{18}$$

Since this example involves only the time domain, Eqs. (11) and (13) reduce to

$$\delta\lambda^{(4)}=\delta u(t_0),\quad \delta\lambda^{(3)}=-\delta\dot{u}(t_0)$$

The variational formulation for this problem is, therefore,

$$\int_0^{t_1}(\dot{u}\delta\dot{u}-\omega^2u\delta u)\mathrm{d}t-\dot{u}(t_1)\delta u(t_1)+v_0\delta u(0)$$

$$+u(0)\delta\dot{u}(0)-u_0\delta\dot{u}(0)=0 \tag{19}$$

Table 3 gives the results for the case $u_0=0$, $v_0=\omega=2\pi$, and $t_1=1$. The assumed shape functions are polynomials in the time variable. A polynomial of order eight again gives good convergence.

Example 3. Response of a Beam to a Moving Mass

A concentrated mass is assumed to move at constant velocity v along the length of a uniform Euler beam, simply supported at each of its ends and having zero displacement and velocity at time $t=0$. Under suitable definitions for k and m, the representative equations may be written[9]:

$$y^{iv}+k\ddot{y}+f(x,t)=0$$

$$y(0,t)=y''(0,t)=y(1,t)=y''(1,t)=0$$

$$y(x,0)=\dot{y}(x,0)=0 \tag{20}$$

The function $f(x,t)$ consists of a sum of inertial terms:

$$f(x,t)=m(\ddot{y}+2v\dot{y}'+g+v^2y'')\delta(x-vt)$$

where g denotes the gravitational constant and δ of the Dirac function. The variational formulation proceeds as previously

Table 3 Solution to free oscillator problem $0\le t\le t_1=1.0$

t	Computed solution	Exact solution
0.0	.00305	0.00000
0.1	.58656	.58779
0.2	.95218	.95106
0.3	.95159	.95106
0.4	.58670	.58779
0.5	−.00058	0.00000
0.6	−.58704	−.58779
0.7	−.95058	−.95106
0.8	−.95147	−.95106
0.9	−.58775	−.58779
1.0	−.00001	0.00000

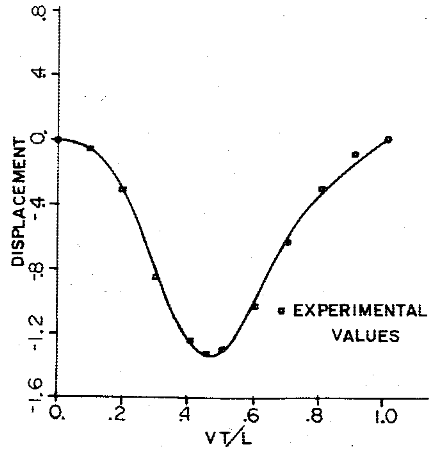

Fig. 1 Displacement of beam at location of moving mass.

indicated:

$$0=\int_0^{t_1}\int_0^{l}[y^{iv}+k\ddot{y}+f(x,t)]\delta y\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^{t_1}\{y(0,t)\delta\lambda_1+y(l,t)\delta\lambda_2+y''(0,t)\delta\lambda_3$$

$$+y''(l,t)\delta\lambda_4\}\mathrm{d}t+\int_0^{l}\{y(x,0)\delta\lambda_5+\dot{y}(x,0)\delta\lambda_6\}\mathrm{d}x \quad (21)$$

In this example, trial functions are assumed such that the boundary conditions are satisfied identically, whereupon the second term in Eq. (21) vanishes identically, leaving only the task of determining $\delta\lambda_5$ and $\delta\lambda_6$. Two partial integrations in space and time, together with substitution of the boundary conditions, yield the following variational statement:

$$0=\int_0^{t_1}\int_0^{l}[y''\delta y''+f(x,t)\delta y+ky\delta\ddot{y}]\mathrm{d}x\,\mathrm{d}t$$

$$+k\int_0^{l}\{[\dot{y}\delta y-y\delta\dot{y}]_0^{t_1}+y(x,0)\delta\lambda_5+\dot{y}(x,0)\delta\lambda_6\}\mathrm{d}x$$

Thus,

$$\delta\lambda_5=-\delta\dot{y}(x,0),\qquad \delta\lambda_6=\delta y(x,0)$$

The appropriate variational equation is, therefore,

$$\int_0^{t_1}\int_0^{l}(y''\delta y''-k\dot{y}\delta\dot{y}+f(x,t)\delta y)\,\mathrm{d}x\,\mathrm{d}t$$

$$+k\int_0^{l}\{\dot{y}(x,t_1)\delta y(x,t_1)-y(x,0)\delta\dot{y}(x,0)\}\mathrm{d}x=0 \quad (23)$$

A matrix approximation to Eq. (23) is obtained as in the first example, again using products of polynomials through order eight. The results are shown in Fig. 1 as a comparison with values scaled from the experimental curves of Ayre, Jacobsen, and Hsu[10] for the case $v=v^*/4$,—v^* being the lowest velocity to produce resonance when the load is a moving weight only. The magnitude assigned to the moving mass is 25% of the total mass of the beam of length L. The displacements have been normalized with respect to the maximum deflection produced if the weight was applied statically at midspan. This problem has also been treated previously by the author,[8] using a conventional finite element method resulting in a set of differential equations in time. The numerical integration of these equations appeared to require a considerably longer computation time.

V. Conclusions

The unconstrained variational statement first developed and used by Tiersten for the solution of field displacements within a body containing a surface of discontinuity can indeed yield solutions to boundary-initial value problems. Further, the variational statement from which such solutions are possible can be formally constructed by an extension of the Lagrange multiplier method in which the unknown multipliers are defined so as to eliminate redundant boundary terms. While the identification of which boundary terms are to be eliminated does not always proceed from obvious physical considerations, this does not appear to impede the application of the procedure.

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